Holomorphic plane fields with many invariant hypersurfaces.

L. Câmara and B. Scárdua

March 27, 2015

Abstract

In this paper we show that a (non necessarily integrable) holomorphic plane field on a compact complex manfold M having an infinite number of invariant hypersurfaces must admit a meromorphic first integral $F : M \longrightarrow \mathbb{C}^p$. In particular, it is integrable.

1 Holomorphic plane fields

Invariant hyperplanes for plane fields. In [7] E. Ghys generalized a result by J.-P. Jouanolou ([8]) showing that any holomorphic Pfaff equation on a compact complex manifold has only a finite number of compact leaves unless all leaves are compact. Both results are generalized versions of a classical result due to G. Darboux (cf. [3],[9]) for holomorphic foliations in dimension two. Our main goal in this paper is to generalize this statement to plane fields of greater codimensions. For this sake, we shall need to revise and introduce some concepts.

Recall from de Medeiros work that a holomorphic plane field of codimension p is locally given by an LDS p-form ([4], Proposition 1.2.1, p. 455), i.e., a germ of holomorphic p-form ω locally decomposable off the singular set $\mathrm{Sing}(\omega)$ with $\mathrm{codim}(\mathrm{Sing}(\omega)) \geq 2$. Thus a holomorphic plane field of codimension p on a complex manifolds M is given by the following data:

1. An open covering $\mathcal{U} = \{U_\alpha\}$ of M;

2. A collection of holomorphic LDS p-forms $\omega_\alpha \in \Omega^p_{\mathcal{O}}(U_\alpha)$ whose singular set $\mathrm{Sing}(\omega)$ has codimension ≥ 2;

3. A collection of maps $g_{\beta\alpha} \in \mathcal{O}^*(U_{\alpha\beta})$ such that $\omega_\beta = g_{\beta\alpha}\omega_\alpha$ whenever $U_{\alpha\beta} = U_\alpha \cap U_\beta \neq \emptyset$.

The Cech cocycle $g = (g_{\beta\alpha}) \in H^1(\mathcal{U}, \mathcal{O}^*)$ defines a line bundle $\mathcal{L}$ over M. Thus the collection (ω_α) may be interpreted as a global holomorphic p-form $\omega = (\omega_\alpha) \in H^0(\mathcal{U}, \Omega^p_{\mathcal{O}} \otimes \mathcal{L})$ defined over M and assuming values on the line

bundle $\mathcal{L}$. The singular set $\mathrm{Sing}(\mathcal{F}_\omega)$ is supposed to be the union of the singular sets of ω_α and has codimension greater or equal to 2.

Recall that the algebraic characterization of the LDS p-forms is given by the equation $i_{v_I}\omega \wedge \omega = 0$ for any p-vector field $v_I = v_1 \wedge \cdots \wedge v_n$, where $\{v_1, \cdots, v_p\}$ is a local frame. Further, the integrability of such plane fields may be characterized by the additional condition $i_{v_I}\omega \wedge d\omega = 0$ ([4], Proposition 1.2.2, p. 455).

A codimension p plane field $\omega = 0$ is said to admit a *first integral* if it is tangent to the the level sets of a holomorphic map $f : M \longrightarrow \mathbb{C}^p$. In such case clearly this plane field is integrable (in the previous sense) and its leaves are contained in the level sets of f. Since in general a compact complex manifold has few holomorphic functions, then we often discuss the meromorphic integrability of such foliations. Clearly such object must be a map $f = (f_1, \cdots, f_p) : M \longrightarrow \mathbb{C}^p$, such that each f_j is a meromorphic function, whose level sets are tangent to the plane field $\omega = 0$. In this case, all the solutions are obviously complete intersections.

Recall that in the codimension 1 case an infinite number of compact solutions leads to the existence of a (global) meromorphic first integral. Thus it is natural to investigate under what conditions on its invariant hypersurfaces a holomorphic plane field of codimension p admits a meromorphic first integral.

Recall that a hypersurface (a codimension 1 variety) is given by an open covering $\mathcal{U} = \{U_\alpha\}$ of M and complex analytic functions $f_\alpha : U_\alpha \longrightarrow \mathbb{C}$ such that $f_\beta = g_{\beta\alpha}f_\alpha$ for some $(g_{\beta\alpha}) \in H^1(\mathcal{U}, \mathcal{O}^*)$. This last condition means essentially that f_α and f_β vanish at the same place in $U_\alpha \cap U_\beta$. A hypersurface $V = (f = 0)$ is said to be *invariant* by ω if $\mathrm{Ker}(\omega(p)) \subset \mathrm{Ker}\, df(p)$ for all $p \in V$. This coincides with the usual geometric definition in case ω is integrable, i.e., the tangent space of the solutions are tangent to V.

Theorem 1 *Let M be a compact complex manifold and $\omega \in H^0(M, \Omega^p \otimes \mathcal{L})$ be induced by LDS p-forms. Then the p-codimension plane field $(\mathcal{F} : \omega = 0)$ admits a meromorphic first integral iff $\mathcal{F}$ admits infinitely many invariant hypersurfaces.*

Chern classes and divisors. Recall that a divisor is a (formal) linear combination of varieties $a_1 f^1 + \cdots + a_k f^k$ representing the variety $(f^1)^{a_1} \cdots (f^k)^{a_k} = 0$, $a_j \in \mathbb{Z}$. This defines a homomorphism $\mathfrak{Div}(M) \longrightarrow \check{H}^1(M, \mathcal{O}^*)$ from the set of analytic divisors in M to the set of line bundles on M. From the exact sequence $0 \longrightarrow \mathbb{Z} \hookrightarrow \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \longrightarrow 0$ one can define the first Chern class $c_1(v) \in \check{H}^2(M, \mathbb{C}) \simeq \check{H}^1(\mathcal{U}, \Omega^1_\mathcal{O})$ of a divisor $v \in \mathfrak{Div}(M)$ as the first Chern class of $(g_{\beta\alpha}) \in \check{H}^1(\mathcal{U}, \mathcal{O}^*)$. Considering the de Rham isomorphism $\check{H}^2(M, \mathbb{C}) \simeq \check{H}^1(\mathcal{U}, \Omega^1_\mathcal{O}) \simeq H^2(M, \mathbb{C})$ one can write the representative of $c_1(v)$ in $\check{H}^1(\mathcal{U}, \Omega^1_\mathcal{O})$ as $d\log g_{\beta\alpha}$. Thus we obtain the homomorphism $\mathfrak{Div}(M) \longrightarrow \check{H}^1(\mathcal{U}, \mathcal{Z}^1_\mathcal{O})$ given by $v \mapsto c_1(v) = d\log g_{\beta\alpha}$, where $(g_{\beta\alpha})$ is the line bundle determined by the divisor v and $\mathcal{Z}^1_\mathcal{O}$ denotes the sheaf of germs of closed holomorphic 1-forms. Now consider the $\mathbb{C}$-linear map $\psi : \mathfrak{Div}(M) \otimes \mathbb{C} \longrightarrow \check{H}^1(\mathcal{U}, \mathcal{Z}^1_\mathcal{O})$ and denote its Kernel by $\mathfrak{Div}_0(M)$. Clearly each $v \in \mathfrak{Div}_0(M)$ is of the form $v = \sum \lambda_j v_j$ where

each $v_j : (f_j = 0)$ is the closure of an invariant hypersurface of $\mathcal{F}$. Note that the linear structure of the map ψ and the existence of infinitely many invariant hypersurfaces ensure the existence of infinitely many invariant divisors in the kernel of ψ, i.e., infinitely many *flat* invariant divisors.

First integrals and p–divisors. Suppose $c_1(v)$ vanishes, then refining the covering if necessary one may find $(\zeta_\alpha) \in \check{H}^0(\mathcal{U}, \mathcal{Z}_\mathcal{O}^1)$ such that $c_1(v) = \zeta_\beta - \zeta_\alpha$. Since with $f_\alpha^j = g_{\alpha\beta}^j \cdot f_\beta^j$, then $\sum_{j=1}^n \lambda_j d \log f_\alpha^j - \lambda_j d \log f_\beta^j = \zeta_\beta - \zeta_\alpha$. We have just proved that

Lemma 1 ([7]) *For each $v = \sum \lambda_j v_j \in \mathfrak{Div}_0$, where v_j is an invariant hypersurface of $\mathcal{F}$ given by $v : (f_j = 0)$, there is a global closed meromorphic 1-form $\xi = (\xi_\alpha) \in \check{H}^0(\mathcal{U}, \mathcal{Z}_\mathcal{M}^1)$ given by $\xi_\alpha = \sum_{j=1}^n \lambda_j d \log f_\alpha^j + \zeta_\alpha$.*

Notice that the global closed meromorphic 1-form $\xi = (\xi_\alpha) \in \check{H}^0(\mathcal{U}, \mathcal{Z}_\mathcal{M}^1)$ is unique up to adding a global closed holomorphic 1-form.

In the sequel we shall need the following technical result well known for 1-forms.

Lemma 2 *Let ω be a holomorphic p-form on $U \subset \mathbb{C}^n$ admitting the hypersurface $V : (f = 0)$ as invariant set. Then there exists a holomorphic $(p + 1)$-form ϖ such that*

$$\omega \wedge d \log(f) = \varpi.$$

Proof. Since $\ker(\omega(x)) \subset \ker(df(x))$ for all $x \in V$, then $df(x) \in \ker(\omega(x))^\perp$ (cf. [5]). From [4], Proposition 1.2.2, one has $df(x) \in \mathcal{E}^*(\omega(x))$ for all $x \in V$. In other words, $\omega(x) \wedge df(x) = 0$ $x \in V$. The result then follows from Rückert nullstellensatz (the analytic version of Hilbert's nullstellensatz). ∎

From Lemmas 1 and 2 we obtain the linear map

$$
\begin{array}{ccc}
\phi : \quad \mathfrak{Div}_0(M) & \longrightarrow & \check{H}^0(\mathcal{U}, \Omega_\mathcal{O}^p \otimes \mathcal{L})/\omega \wedge \check{H}^0(\mathcal{U}, \mathcal{Z}_\mathcal{O}^2) \\
\left(v = \sum_{j=1}^n \lambda_j v_j \right) & \mapsto & [\omega \wedge \xi]
\end{array}
$$

where $\mathfrak{Div}_0$ denotes the set of flat divisors (i.e., those with vanishing first Chern class).

Since ϕ is linear, $\dim(\check{H}^0(\mathcal{U}, \Omega_\mathcal{O}^p \otimes \mathcal{L}))$ is finite, and, by hypothesis, $\mathfrak{Div}_0(M)$ has infinite dimension, we may obtain p linearly independent divisors (v_i), $i = 1, \cdots, p$, in the kernel of ϕ. Thus we have a closed meromorphic 1-form ξ such that $\xi \wedge \omega = 0$. Repeating this process, we may find linearly independent closed meromorphic 1-forms $\xi_{i_1}, \cdots, \xi_{i_p}$ as above. Thus $\xi_I = \xi_{i_1} \wedge \cdots \wedge \xi_{i_p}$ is a closed meromorphic p-form. Let $\eta_I = \xi_{i_1} \wedge \cdots \wedge \xi_{i_p}$, since it is decomposable, then $\mathcal{E}^*(\eta_I) = \mathcal{I}(\xi_{i_1}, \cdots, \xi_{i_p})$, thus $\mathcal{E}^*(\eta_I(x)) \subset \mathcal{E}^*(\omega(x))$ for all x off the polar set of η_I, say $P(\eta_I)$. From de Medeiros division Lemma ([4], Lemma 3.1.1, p. 460) there are functions h_I holomorphic off $P(\eta_I)$ such that $\omega = h_I \eta_I$. As a consequence, there are global meromorphic functions h_{IJ} such that $\xi_I = h_{IJ} \xi_J$. Since ξ_I is closed, then $0 = dh_{IJ} \wedge \xi_J$. In particular, $0 = dh_{IJ} \wedge \omega$. Choosing the I and J properly (since $\mathfrak{Div}_0(M)$ has infinite dimension), we may find the desired global meromorphic first integral $h : M \longrightarrow \mathbb{C}^p$.

3

2 Transversely holomorphic structures

The notion of a smooth manifold endowed with a transversely holomorphic structure was introduced in the work of X. Gomez-Mont ([6]). Behind all the construction is the notion of structural sheaf. More precisely, we suppose the existence of a smooth manifold M admitting a real smooth foliation $\mathcal{F}$ of real codimension $2p$ with holomorphic transversal sections and transversely holomorphic transition maps. The *structural sheaf* $\mathcal{O}$ being the germ of functions constant along the leaves and transversely holomorphic. The structural sheaf is also called the sheaf of *basic transversely holomorphic functions*. In a similar fashion one may define the sheaf of *basic transversely meromorphic functions* $\mathcal{M}$, *basic transversely holomorphic p-forms* $\Omega_{\mathcal{O}}^p$, the *basic transversely meromorphic p-forms* $\Omega_{\mathcal{M}}^p$ etc.

Therefore, one may construct structures similar to the ones present in the classical holomorphic case like divisors and associate line bundles. Using as key point the finiteness of $\dim_{\mathbb{C}}(\check{H}^0(\mathcal{U}, \Omega_{\mathcal{O}}^{p+1} \otimes \mathcal{L}))$ (cf. [6]), Brunella and Nicolau ([1]) repeated essentially the same arguing in ([7]) in order to prove the existence of non constant basic transversely meromorphic functions for any transversely holomorphic structure of complex codimension 1 defined on a compact and connected manifold M admitting infinitely many compact solutions.

Since Gomez-Mont theory is constructed for transversely holomorphic structures of any complex codimension, it is natural to study the same problem for greater codimensions. Again we need to introduce some notation. The first remark is that we are dealing with foliations without singularities, thus a transversely holomorphic foliation of complex codimension p on a complex manifolds M is given by the following data:

1. An open covering $\mathcal{U} = \{U_\alpha\}$ of M;

2. A collection of non-singular, basic transversely holomorphic, locally decomposable, and integrable p-forms $\omega_\alpha \in \Omega_{\mathcal{O}}^p(U_\alpha)$;

3. A collection of maps $g_{\beta\alpha} \in \mathcal{O}^*(U_{\alpha\beta})$ such that $\omega_\beta = g_{\beta\alpha}\omega_\alpha$ whenever $U_{\alpha\beta} = U_\alpha \cap U_\beta \neq \emptyset$.

Again the Cech cocycle $g = (g_{\beta\alpha}) \in H^1(\mathcal{U}, \mathcal{O}^*)$ defines a line bundle $\mathcal{L}$ over M. Thus the collection (ω_α) may be interpreted as a global basic transversely holomorphic p-form $\omega = (\omega_\alpha) \in H^0(\mathcal{U}, \Omega_{\mathcal{O}}^p \otimes \mathcal{L})$ defined over M and assuming values on the line bundle $\mathcal{L}$.

A transversely holomorphic hypersurface (a codimension 1 variety) is given by an open (trivializing) covering $\mathcal{U} = \{U_\alpha\}$ of M and complex functions $f_\alpha : U_\alpha \longrightarrow \mathbb{C}$ such that $f_\beta = g_{\beta\alpha}f_\alpha$ for some $(g_{\beta\alpha}) \in H^1(\mathcal{U}, \mathcal{O}^*)$. This last condition means essentially that f_α and f_β vanish at the same place in $U_\alpha \cap U_\beta$ with transversely holomorphic structure. Notice that $f = (f_\alpha)$ is not necessarily basic. A hypersurface $V : (f = 0)$ is said to be *invariant* by ω if $\mathrm{Ker}(\omega(p)) \subset \mathrm{Ker}\, df(p)$ for all $p \in V$. This coincides with the usual geometric definition in case ω is integrable, i.e., the tangent space of the solutions are tangent to V.

Again a linear combination with complex coefficients $v = a_1 v_1 + \cdots + a_n v_n$ is a *divisor* if each $a_j \in \mathbb{Z}$ and each v_j is a transversely holomorphic hypersurface representing the variety $(f^1)^{a_1} \cdots (f^k)^{a_k} = 0$. In a similar fashion, we obtain the linear map $\psi : \mathfrak{Div}(M) \otimes \mathbb{C} \longrightarrow \check{H}^1(\mathcal{U}, \Omega_{\mathcal{O}}^1)$ given by $v = \sum_{j=1}^n \lambda_j v_j \mapsto \sum_{j=1}^n \lambda_j c_1(v_j) = \sum_{j=1}^n \lambda_j d \log g_{\beta\alpha}^j$, where $(g_{\beta\alpha}^j)$ is the line bundle determined by the invariant hypersurface v_j.

Once again the linear structure of the map ψ and the existence of infinitely many invariant transversely holomorphic hypersurfaces ensure the kernel of ψ has sinfinite dimension, i.e., the $\mathbb{C}$-linear space $\mathfrak{Div}_0(M)$ of *flat* invariant divisors has infinite dmension. From adapted versions of Lemmas 1 and 2 we obtain the following result.

Theorem 2 *Let M be a smooth manifold endowed with a transverse holomorphic structure of complex codimension p given by $\omega \in H^0(M, \Omega^p \otimes \mathcal{L})$. Then the p-codimension plane field $(\mathcal{F} : \omega = 0)$ admits a basic transversely meromorphic first integral $F : M \longrightarrow \mathbb{C}^p$ iff $\mathcal{F}$ admits infinitely many invariant transversely holomorphic hypersurfaces.*

The proof is essentially the same as the one for the holomorphic case. In fact it is even simpler due to the absence of singularities. For the smooth version of de Medeiros division Lemma is almost immediate.

References

[1] M. Brunella & M. Nicolau, *Sur les hypersurfaces solutions des équations de Pfaff*. C. R. Acad. Sci. Paris Sér. I Math. 329 (1999), no. 9, 793–795.

[2] E. Cartan, *Leçons sur les invariants intégraux*. Paris, Hermann, 1922.

[3] G. Darboux, *Mémoire sur les équations différentielles algébriques du second ordre et du premier degré*. Bulletin des sciences mathématiques et astronomiques, Sér. 2, 2 no. 1 (1878), pp. 60-96, 123-144, 151-200.

[4] A.S. de Medeiros, *Singular foliations and differential p-forms*, Ann. Fac. Sci. Toulouse Math. (6) 9 (2000), no. 3, 451–466.

[5] C. Godbillon, *Geométrie Différentielle et Mécanique Analytique*, Hermann Paris 1969.

[6] X. Gomez-Mont, *Transversal holomorphic structures*, J. Differ. Geom. 15 (1980) 161-185.

[7] E. Ghys. *À propos d'un théoréme de J.-P. Jouanolou concernant les feuilles fermées des feuilletages holomorphes*. Rend. Circ. Mat. Palermo (2) 49 (2000), no. 1, 175–180.

[8] J.-P. Jouanolou, *Hypersurfaces solutions d'une équation de Pfaff analytique*. Math. Ann., (1978), 239-245.

[9] J.-P. Jouanolou, *Équations de Pfaff algébriques*, Lecture Notes in Math. 708, Springer-Verlag, Berlin, 1979.